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QUINTIC SPLINE SOLUTIONS OF BOUNDARY VALUE PROBLEMS

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Abstract—A new fourth order method using quintic polynomials is designed in this paper for the smooth approximation of the two point boundary value problems involving second order differential equations lacking the first derivative. The present method enables us to approximate the unknown function as well as its derivative at every point of the range of integration and thus it has obvious advantages over other discrete numerical methods. Our present method outperforms the well-known fourth order Noumerov's finite difference scheme. The convergence of the method is briefly outlined using matrix algebra and two numerical illustrations are provided to demonstrate the practical suitability of our approach.

1. DEFINITION OF THE PROBLEM

We consider the two point boundary value problem

$$y''(x) = F(x)y(x) + g(x),\ y(a) - A_1 = y(b) - A_2 = 0 \tag{1.1}$$

where $F(x)$ and $g(x)$ are continuous functions with $F(x) \geq 0$ on $[a, b]$ and a, b, A_1, A_2 are arbitrary real finite constants. The problems of the form (1.1) frequently arise in theory of structure and a variety of other scientific applications. The analytical solutions of (1.1) is not obtained in general for arbitrary choices of $F(x)$ and $g(x)$. We usually resort to some numerical technique for an approximate solution of (1.1). A more commonly used finite difference method for solving (1.1) numerically is discussed by many authors and we refer the reader in particular to Fox[1], Henrici[2], Aziz *et al.*[3], Bramble *et al.*[4], Fischer *et al.*[5] and Usmani[6].

The possibility of using spline functions for obtaining a smooth approximate solution of (1.1) is briefly discussed by Ahlberg *et al.*[7]. Following this Bickley[8] and Albasiny *et al.*[9] have demonstrated the use of cubic spline functions for obtaining an approximate solution of (1.1). Albasiny *et al.* have in particular emphasized a connection that exists between a cubic spline solution and a solution obtained by standard finite difference Noumerov's formula.

In this brief note we demonstrate the use of quintic spline functions for obtaining an approximate solution of (1.1).

2. DEVELOPMENT OF THE FORMULAS

Divide $[a, b]$ into $N + 1$ equal parts and thus define the sequence x_n

$$x_n = a + nh,\ n = 0(1)\overline{N+1}$$

with $h = (b - a)/(N + 1)$, $N \geq 3$. Let $y(x)$ be the exact solution of the system (1.1) and y_i be an approximation to $y(x_i)$ obtained by the quintic polynomial $P_i(x)$ passing through the points (x_i, y_i) and (x_{i+1}, y_{i+1}), $i = 0(1)N$,

$$P_i(x) = a_i(x - x_i)^5 + b_i(x - x_i)^4 + c_i(x - x_i)^3 + d_i(x - x_i)^2 + e_i(x - x_i) + f_i. \tag{2.1}$$

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we designate

$$P_i''(x_j) = M_j,\ P_i^{(4)}(x_j) = S_j,\ j = i, i+1,\ i = 0(1)N. \tag{2.2}$$

Note that M_j and S_j are approximations to $y''(x_j)$ and $y^{(4)}(x_j)$ respectively.
From the conditions (2.2) for $j = i$, $i+1$ and

$$P_i(x_i) = y_i,\ y_i(x_{i+1}) = y_{i+1} \tag{2.3}$$

we determine the six coefficients in (2.1) as functions of y_i, y_{i+1}, M_i, M_{i+1}, S_i, S_{i+1} in the form

$$a_i = (S_{i+1} - S_i)/(120h),\ b_i = S_i/24$$

$$c_i = (M_{i+1} - M_i)/(6h) - h(S_{i+1} + S_i)/36,\ d_i = M_i/2$$

$$e_i = (y_{i+1} - y_i)/h - h(M_{i+1} + 2M_i)/6 + h^3(7S_{i+1} + 8S_i)/360$$

$$f_i = y_i,\ i = 0(1)N \tag{2.4}$$

with $y_0 = A_1$, $y_{N+1} = A_2$, $M_i \simeq F_i y_i + g_i$, $i = 0(1)\overline{N+1}$, and $F_i \equiv F(x_i)$, etc. These formulas are also given for the grid points not equally spaced by Späth[10] (p. 90). Now from the continuity of the first derivatives namely

$$P_j'(x_{j+1}) = P_{j+1}'(x_{j+1}),\ j = i-2, i-1, i$$

we have the relations

$$7S_{j+1} + 16S_j + 7S_{j-1} = 60(M_{j+1} + 4M_j + M_{j-1})/h^2 - 360(y_{j+1} - 2y_j + y_{j-1})/h^4,$$
$$j = i-1, i, i+1. \tag{2.5}$$

Similarly from the continuity of the third derivatives, we have

$$S_{j+1} + 4S_j + S_{j-1} = 6(M_{j+1} - 2M_j + M_{j-1})/h^2,\ j = i-1, i, i+1. \tag{2.6}$$

Multiply (2.6) by 7 and from the resulting equation subtract (2.5) to obtain

$$h^4 S_j = 30(y_{j+1} - 2y_j + y_{j-1}) - 3h^2(M_{j+1} + 18M_j + M_{j-1})/2. \tag{2.7}$$

Substitute from this, S_j, $j = \overline{i-1}(1)\overline{i+1}$ either in (2.5) or (2.6) to get

$$y_{i-2} + 2y_{i-1} - 6y_i + 2y_{i+1} + y_{i+2} = h^2(M_{i-2} + 26M_{i-1} + 66M_i + 26M_{i+1} + M_{i+2})/20, \tag{2.8}$$

$i = 2(1)\overline{N-1}$, where $M_i \simeq y_i'' = F_i y_i + g_i$. This recurrence relation gives $(N-2)$ linear equations in the unknowns y_i, $i = 1(1)N$.

In an analogous manner we develop two more recurrence relations, namely,

$$(i)\qquad 4y_0 - 7y_1 + 2y_2 + y_3 = h^2[4M_0 + 41M_1 + 14M_2 + M_3]/12 \tag{2.9}$$

$$(ii)\qquad y_{N-2} + 2y_{N-1} - 7y_N + 4y_{N+1} = h^2[M_{N-2} + 14M_{N-1} + 41M_N + 4M_{N+1}]/12$$

using quartic splines approximating $y(x)$ in the intervals $[x_i, x_{i+1}]$, $i = 0, 1, 2$ and $i = N-2$, $N-1$, N. The determination of N unknowns y_i, $i = 1(1)N$ can now be effected by solving the set of N linear equations given by 2.9(i), (2.8) and 2.9(ii). The knowledge of y_i, $i = 0(1)\overline{N+1}$ now enables us to compute $M_j = y_j'' = F_j y_j + g_j$, $j = 0(1)\overline{N+1}$. From (2.7) we now compute S_j, $j = 1(1)N$. We can now compute S_0 and S_{N+1} from the relations

$$S_0 = [48(-2y_0 + 5y_1 - 4y_2 + y_3 + h_2M_0)/h^4 - 21S_1 - 12S_2 - S_3]/10, n = 3$$

$$= 120(y_0 - 4y_1 + 6y_2 - 4y_3 + y_4)/h^4 - 26S_1 - 66S_2 - 26S_3 - S_4, N > 3 \quad (2.10)$$

and similar expressions for S_{N+1}, see Appendix B. The knowledge of y_i, M_i, S_i, $i = 0(1)\overline{N+1}$ enables us to write down the coefficients in (2.1) as given by (2.4). Finally we have y_i', $i = 0(1)\overline{N+1}$ in the form

$$y_i' = e_i, \; i = 0(1)N, \text{ and}$$

$$y_{N+1}' = 5h^4A_N + 4h^3B_N + 3h^2C_N + 2hD_N + e_N. \quad (2.11)$$

In conclusion we remark that the schemes (2.9) can also be developed in an alternative manner. Write the Noumerov's formula [see(4.3)] for $n = 1$ and $n = 2$. We thus have

(i) $$y_0 - 2y_1 + y_2 = h^2(y_0'' + 10y_1'' + y_2'')/12, \text{ and}$$

(ii) $$y_1 - 2y_2 + y_3 = h^2(y_1'' + 10y_2'' + y_3'')/12. \quad (2.12)$$

Now 2.9(i) follows on multiplying 2.12(i) by 4 and adding the resulting scheme to 2.12(ii). In a similar manner we can derive 2.9(ii).

3. MATRIX FORMULATION OF THE PROBLEM

Let $Y = (y_i)$, $d = (d_i)$ be N dimensional column vectors and $A = (a_{ij})$ be an $(N \times N)$ matrix such that

$$d_1 = (4 - h^2F_0/3)A_1 - h^2(4g_0 + 41g_1 + 14g_2 + g_3)/12$$

$$d_2 = (1 - h^2F_0/20)A_1 - h^2(g_0 + 26g_1 + 66g_2 + 26g_3 + g_4)/20$$

$$d_i = -h^2(g_{i-2} + 26g_{i-1} + 66g_i + 26g_{i+1} + g_{i+2})/20, \; i = 3(1)\overline{N-2}$$

$$d_{N-1} = (1 - h^2F_{N+1}/20)A_2 - h^2(g_{N-3} + 26g_{N-2} + 66g_{N-1} + 26g_N + g_{N+1})/20$$

$$d_N = (4 - h^2F_{N+1}/3)A_2 - h^2(g_{N-2} + 14g_{N-1} + 41g_N + 4g_{N+1})/12 \quad (3.1)$$

and

$$a_{12} = -2 + 7h^2F_2/6, \; a_{N,N-1} = -2 + 7h^2F_{N-1}/6$$

$$a_{13} = -1 + h^2F_3/12, \; a_{N,N-2} = -1 + h^2F_{N-2}/12$$

$$a_{ij} = 7 + 41h^2F_j/12, \; i = j = 1, N$$

$$= 6 + 33h^2F_j/10, \; i = j = 2(1)\overline{N-1}$$

$$= -2 + 13h^2F_j/10, \; |i-j| = 1; \; i, j = 2(1)\overline{N-1}$$

$$= -1 + h^2F_j/20, \; |i-j| = 2; \; i, j = 3(1)\overline{N-2}$$

$$= 0, \; |i-j| > 2. \quad (3.2)$$

It is easily seen that the unknown vector Y satisfies the matrix equation

$$AY = d \quad (3.3)$$

This system of linear equations is solved for Y by a modified Gaussian elimination algorithm the details of which are not included here, see Späth[10]. If $\bar{Y} = (y(x_i))$, then we can easily show that $\bar{Y}$ satisfies

$$A\bar{Y} = d + t, \tag{3.4}$$

where the vector $t = (t_i)$ has components

$$t_i = \begin{cases} -h^6 y^{(6)}(w_1)/48, x_0 < w_1 < x_3 \\ h^6 y^{(6)}(x_i)/120 + 0(h^7), i = 2(1)\overline{N-1} \\ -h^6 y^{(6)}(w_N)/48, x_{N-2} < w_N < x_{N+1}. \end{cases} \tag{3.5}$$

For the derivation of local truncation errors we follow Henrici[2].

On subtracting (3.3) from (3.4) we derive the error equation

$$AE = t \tag{3.6}$$

where $E = (\sigma_i)$ is the error vector and the error of discretization is defined by $\sigma_i = y(x_i) - y_i$. Thus σ_i is the amount by which the numerical approximation y_i deviates from the actual solution $y(x_i)$ of (1.1) at $x = x_i$. We can establish that $\|E\| = \max_i|\sigma_i| = 0(h^4)$ using the theory of monotone matrices given by Henrici[2], see Appendix A. The numerical method derived by Albasiny *et al.*[9] using cubic splines is a 2nd order convergent process although the authors have not proved this. We shall now present a survey of our numerical experiments in the next section.

4. NUMERICAL ILLUSTRATIONS

We choose the following examples for experimentation.

$$x^2 y'' = 2y - x, y(2) = y(3) = 0 \tag{4.1}$$

with $y(x) = (19x - 5x^2 - 36x^{-1})/38$.

$$y'' = y - 4x \exp(x), y(0) = y(1) = 0 \tag{4.2}$$

with $y(x) = x(1-x)\exp(x)$. The numerical calculations were made on an IBM 370 computer using double precision arithmetic in order to reduce the roundoff errors to a minimum.

For the sake of comparison we also solved these problems by Noumerov's method in the unknowns y_n, $n = 1(1)N$, satisfying

$$y_{n-1} - 2y_n + y_{n+1} = h^2[y''_{n-1} + 10y''_n + y''_{n+1}]/12, n = 1(1)N. \tag{4.3}$$

It is proved in [2] that $\|E\| = 0(h^4)$ based on (4.3). We remark that our method outperforms also a fourth order method of Bramble and Hubbard[4] and it gives better results than a fourth order method of Aziz and Hubbard[3] whenever $g(x)$ in (1.1) is a constant. Their method is characterised by the difference scheme (that is y_i, $i = 1(1)N$, satisfy)

$$\begin{aligned}(-1 + h^2F_{n-1}/12)y_{n-1} + (2 + 10h^2F_n/12)y_n + (-1 + h^2F_{n+1}/12)y_{n+1} \\ = -h^2 g_n + h^4 g''_i/12, i = 1(1)N.\end{aligned} \tag{4.4}$$

In the general case, however, Aziz and Hubbard method gives somewhat better results than our method. In Table 1 we compare these methods with different step-sizes for the problem (4.2).

Table 1.

h	E_0 = observed max $\lvert\sigma_i\rvert$		
	Noumerov	Our method	Aziz and Hubbard
1/4	0.93E–4†	0.55E–4	0.12E–5
1/8	0.59E–5	0.11E–5	0.81E–7
1/16	0.37E–6	0.11E–6	0.53E–8
1/32	0.23E–7	0.75E–8	0.33E–9

†We write 0.93E–4 for 0.93×10^{-4}.

We have not included the results based on the method of Bramble and Hubbard which performs poorly even in relation to Noumerov's method. In case, if $g(x)$ is constant in (1.1), then the two methods of Noumerov and that of Aziz and Hubbard give identical results. Also these other finite difference methods do not produce y_i', $i = 0(1)N + 1$ as our method does. We now summarize the results on our present method for both problems in Table 2. Our experimental results in Table 2 confirm that on halving the step-size h, the quantity E_0 is reduced approximately by a factor of $2^{-4} = 0.0625$ for small values of h, except, of course, for cases when rounding errors are significant.

We have tabulated the coefficients of quintic splines (2.1) for the problem (4.1) in Table 3. We observe that max $|\sigma_i| < 0.182\text{E–}5$, max $|y'(x_i) - y_i| < 0.221\text{E–}3$, max $|y''(x_i) - y_i''| < 0.717\text{E–}6$ and max $|y^{(4)}(x_i) - y_i^{(4)}| < 0.180\text{E–}1$. Usually the spline approximations for higher derivatives of $y(x)$ in the outermost part of the range of integration are poor. This results from the poor accuracy of S_i provided by (2.7) and (2.10). If in (1.1) $F(x)$ happens to be a constant, say $F(x) \equiv c$, then we may not use these equations and compute S_i from

$$S_i = c^2 y_i + c g_i + g_i'', \; i = 0(1)\overline{N+1}.$$

5. CONCLUSION

The present method has advantages such as increased accuracy, a continuous approximation $P_i(x)$ to $y(x)$ in each interval $[x_i, x_{i+1}]$ instead of values at isolated grid points. The order of converge of the method is 4 and outperforms Noumerov's finite difference method of the same order.

Table 2. Observed maximum absolute error $E_0 = \max|\sigma_i|$

h	Problem (4.1)	Problem (4.2)
1/4	0.182E–5	0.547E–4
1/8	0.313E–7	0.110E–5
1/16	0.292E–8	0.109E–6
1/32	0.216E–9	0.751E–8
1/64	0.141E–10	0.481E–9
1/128	0.891E–12	0.303E–10
1/256	0.591E–13	0.185–11
1/512	0.176E–13	0.648–12
1/1024	0.571E–13	0.764–12

Table 3. $h = 1/4$

i	0	1	2	3	4
a_i	0.546E–2	0.651E–2	0.212E–2	0.317E–2	
$b_i = y_i^{(4)}/24$	−0.0240	−0.0171	−0.0090	−0.0063	
$c_i = y_i^{(3)}/6$	0.538E–1	0.344E–1	0.226E–1	0.154E–1	
$d_i = y_i''/2$	−0.250	−0.214	−0.192	−0.177	−0.166
$e_i = y_i'$	0.211	0.950E–1	−0.631E–2	−0.984E–1	−0.184
$f_i = y_i$	0.0	0.3783E–1	0.4868E–1	0.3544E–1	0.0

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APPENDIX A

Analysis of the discretization error and convergence of the method

This analysis depends on the properties of the matrices A and A_0 where A_0 is a five band matrix obtained from A by setting each $f_i \equiv 0$, so that

$$A_0 = \begin{bmatrix} 7 & -2 & -1 & & & & \\ -2 & 6 & -2 & -1 & & & \\ -1 & -2 & 6 & -2 & -1 & & \\ & & \cdots & \cdots & \cdots & & \\ & -1 & -2 & 6 & -2 & -1 \\ & & -1 & -2 & 6 & -2 \\ & & & -1 & -2 & 7 \end{bmatrix}. \tag{A1}$$

From the theory of monotone matrices[2] it follows that both the matrices A and A_0 are monotone satisfying $A \geq A_0$ and hence $A^{-1} \leq A_0^{-1}$ provided

$$13h^2F_M < 20, \quad F_M = \max_{a \leq x \leq b} f(x). \tag{A2}$$

Furthermore it can be verified

$$A_0 = PQ \tag{A3}$$

where $P = (p_{ij})$, $Q = (q_{ij})$ are tridiagonal matrices with $p_{ii} = 2$, $p_{ij} = -1$, $|i-j| = 1$; $q_{ii} = 4$, $q_{ij} = 1$, $|i-j| = 1$. We can also verify that

$$A_0^{-1} = [P^{-1} + Q^{-1}]/6, \tag{A4}$$

and

$$\begin{aligned} \|A_0^{-1}\| &\leq [\|P^{-1}\| + \|Q^{-1}\|]/6 \\ &\leq \frac{1}{6}\left[(b-a)^2/(8h^2) + \left(1 - \operatorname{sech}\frac{N+1}{2}\theta\right)\Big/2\right], \cosh\theta = 2 \\ &\leq [b-a)^2/8h^2) + 1/2]/6. \end{aligned} \tag{A5}$$

For the norms see Ref.[5]. Now rewriting (3.6) in the form $E = A^{-1}T$ or $|E| < A_0^{-1}|T|$, it follows by (A5) that

$$\|E\| \leq h^4(b-a)^2M_6/2304 + 0(h^6) = 0(h^4), \; M_6 = \max_x |y^{(6)}| \tag{A6}$$

The inequality (A6) proves that our numerical procedure is a fourth order convergent process.

In proving (A6), we have used $\|T\| < h^6M_6/48$ which follows from (3.5) on noting that

$$t_i = h^6 \int_{-2}^{2} G(s) y^{(6)}(x_i + hs)\, ds, \; i = 2(1)\overline{N-1},$$

where $G(s) = G(-s)$

$= (2-s)^5/5! - (2-s)^3/120, 1 \le s \le 2$

$= [(2-s)^5 + 2(1-s)^5]/5! - [(2-s)^3 + 26(1-s)^3]/120, 0 \le s \le 1.$

It can easily be seen that $G(s) \ge 0$ on $[0, 1]$ and $G(s) \le 0$ on $[1, 2]$,

$$|t_i| \le h^6 M_6 \int_{-2}^{2} |G(s)|\, ds < 0.011113 \ldots h^6 M_6,$$

$$< h^6 M_6/48, \; i = 2(1)\overline{N-1} \quad \text{and thus finally}$$

$$|t_i| < h^6 M_6/48, \; i = 1(1)N \text{ and hence } \|T\| < h^6 M_6/48.$$

Remark. For a given vector $v = (v_i)$, we have $\|v\| = \max_i |v_i|$ and for a matrix

$$A = (a_{ij}), \|A\| = \max_i \sum_j |a_{ij}|.$$

APPENDIX B

Derivation of (2.10)

We can eliminate M_j, $j = \overline{i-2}\ (1)\ \overline{i+2}$, from the six relations given by (2.5) and (2.6) to obtain

$$y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2} = h^4(S_{i-2} + 26S_{i-1} + 66S_i$$

$$+ 26S_{i+1} + S_{i+2})/120, \quad i = 2(1)\overline{N-1}. \tag{B1}$$

This relation for $i = 2$ and $N - 1$ gives S_0 and S_{N+1} respectively for $N > 3$ as given by (2.10).

For $N = 3$, we develop in an analogous manner the recurrence relations

$$-2y_0 + 5y_1 - 4y_2 + y_3 = -h^2 M_0 + h^4(10S_0 + 21S_1 + 12S_2 + S_3)/48, \tag{B2}$$

and

$$y_1 - 4y_2 + 5y_3 - 2y_4 = -h^2 M_4 + h^4(S_1 + 12S_2 + 21S_3 + 10S_4)/48, \tag{B3}$$

which can be solved to produce S_0 or S_4 as given by (2.10).